MAE 606

HW # 8

We are given the direction of the equilibrium angular velocity, $\hat{\boldsymbol{v}} = \boldsymbol{b}_3$. So, computing the equilibrium angular-velocity vector for this problem requires computing only the magnitude Ω. We do so by equating the angular momentum for the initial conditions ($\boldsymbol{H}_0$) to the angular momentum in equilibrium ($\boldsymbol{H}_e$). Note that while these vectors are equal, their representations in B are not. To be precise, ${}^{\mathrm{B}}H_0 \neq {}^{\mathrm{B}}H_e$. So, it is simplest (and sufficient) to equate only their magnitudes.

We are given $\boldsymbol{I}_c$ and $\boldsymbol{I}_d$. Furthermore, as we derived in class, we can lump together the wheel inertia dyadics (whatever they are) with that of the body as a single constant. I.e., we define the sum

$$\boldsymbol{I}_{b+w} = \boldsymbol{I}_b + \sum_{i=1}^{4} \boldsymbol{I}_{wi} = \boldsymbol{I}_c - \boldsymbol{I}_d .$$

Initially the angular velocity of the body is along the principal axis of I_c nearest $\boldsymbol{b}_3$. Performing an eigenvector decomposition of I_c provides the needed ${}^{\mathrm{B}}p_3$. For example, use `[V,D]=eig(Ic);` in MATLAB. With this information we can now write $\boldsymbol{H}_0$ explicitly:

$${}^{\mathrm{B}}H_0 = {}^{\mathrm{B}}I_{b+w}\, {}^{\mathrm{B}}\omega^{\mathrm{B/N}}(0) + {}^{\mathrm{B}}I_d\, {}^{\mathrm{B}}\omega^{\mathrm{D/N}}(0) + {}^{\mathrm{B}}h$$

Substituting ${}^{\mathrm{B}}\omega^{\mathrm{D/N}}(0) = {}^{\mathrm{B}}\omega^{\mathrm{B/N}}(0)$ and ${}^{\mathrm{B}}\omega^{\mathrm{B/N}}(0) = 0.5\,{}^{\mathrm{B}}p_3$ leads to

$${}^{\mathrm{B}}H_0 = 0.5\left({}^{\mathrm{B}}I_{b+w} + {}^{\mathrm{B}}I_d\right){}^{\mathrm{B}}p_3 + {}^{\mathrm{B}}h$$

$${}^{\mathrm{B}}H_0 = 0.5\,{}^{\mathrm{B}}I_c\, {}^{\mathrm{B}}p_3 + {}^{\mathrm{B}}h$$

In equilibrium spin about $\boldsymbol{b}_3$, dynamic balance requires

$${}^{\mathrm{B}}h = {}^{\mathrm{B}}b_3^{\times}\, {}^{\mathrm{B}}b_3^{\times} I_{b+w} \Omega\, {}^{\mathrm{B}}b_3 .$$

Specifically,

$${}^{\mathrm{B}}h = -\Omega \begin{bmatrix} -30 \\ 20 \\ 0 \end{bmatrix}.$$

For generality we will use ${}^{\mathrm{B}}I_{13}$ instead of –30 and ${}^{\mathrm{B}}I_{23}$ instead of 20 for the rest of this derivation. Using the fact that ${}^{\mathrm{B}}I_c\, {}^{\mathrm{B}}p_3 = {}^{\mathrm{P}}I_3\, {}^{\mathrm{B}}p_3$, where ${}^{\mathrm{P}}I_3$ is the principal moment of inertia for $\boldsymbol{p}_3$ (i.e. an ${}^{\mathrm{P}}I_3$ is an eigenvalue of ${}^{\mathrm{B}}I_c$) the initial angular momentum is

$$ {}^{B}H_0 = 0.5\,{}^{P}I_3\,{}^{B}p_3 - \Omega\begin{bmatrix} {}^{B}I_{13} \\ {}^{B}I_{23} \\ 0 \end{bmatrix}. $$

The final angular momentum is

$$ {}^{B}H_e = {}^{B}I_c\begin{bmatrix} 0 \\ 0 \\ \Omega \end{bmatrix} - \Omega\begin{bmatrix} {}^{B}I_{13} \\ {}^{B}I_{23} \\ 0 \end{bmatrix} = \Omega\,{}^{B}I_{33}, $$

where ${}^{B}I_{33}$ is the moment of inertia of $\boldsymbol{I}_c$ about $\boldsymbol{b}_3$ (not the principal moment of inertia because the $\boldsymbol{p}$ axes are not aligned with the $\boldsymbol{b}$ axes). Equating the magnitudes starts with

$$ {}^{B}H_e^2 = {}^{B}H_0^2 $$

$$ \Omega^2\,{}^{B}I_{33}^2 = 0.25\,{}^{P}I_3^2 - 2\Omega\,{}^{B}p_3^{T}\begin{bmatrix} {}^{B}I_{13} \\ {}^{B}I_{23} \\ 0 \end{bmatrix} + \Omega^2\left({}^{B}I_{13}^2 + {}^{B}I_{23}^2\right). $$

This equation is quadratic in Ω:

$$ \left({}^{B}I_{13}^2 + {}^{B}I_{23}^2 - {}^{B}I_{33}^2\right)\Omega^2 - 2\,{}^{B}p_3^{T}\begin{bmatrix} I_{13} \\ I_{23} \\ 0 \end{bmatrix}\Omega + 0.25\,{}^{P}I_3^2 = 0. $$

Therefore, we can solve for Ω with the quadratic formula:

$$ \Omega = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}, $$

where

$$ a = \left({}^{B}I_{13}^2 + {}^{B}I_{23}^2 - {}^{B}I_{33}^2\right) $$

$$ b = 2\,{}^{B}p_3^{T}\begin{bmatrix} I_{13} \\ I_{23} \\ 0 \end{bmatrix} $$

$$ c = 0.25\,{}^{P}I_3^2 $$

We use the positive root because the negative one corresponds to backward spin (or spin about $-\boldsymbol{b}_3$, which is not what the problem statement asks for).

The wheel momentum in equilibrium is found by using this value of Ω in

$$ {}^{B}h = -\Omega \begin{bmatrix} -30 \\ 20 \\ 0 \end{bmatrix}. $$

The wheel spin axes are given. Furthermore, the wheel inertia for the spin axis is found from the given information with $I_{wi} = \dfrac{70}{6000 \times \dfrac{\pi}{30}}$. To find the wheel speeds in equilibrium we can pseudoinvert the wheel-actuator Jacobian and multiply it by this wheel momentum. Ordered and stacked in a matrix *A*, the spin axes form the Jacobian

$$ A = \begin{bmatrix} {}^{B}\hat{a}_1 & {}^{B}\hat{a}_2 & {}^{B}\hat{a}_3 & {}^{B}\hat{a}_4 \end{bmatrix} $$

$$ A = \begin{bmatrix} 1 & 0 & 0 & \sqrt{3}/3 \\ 0 & 1 & 0 & \sqrt{3}/3 \\ 0 & 0 & 1 & \sqrt{3}/3 \end{bmatrix} $$

Then the Moore-Penrose pseudoinverse is

$$ A^{+} = \frac{1}{6} \begin{bmatrix} 5 & -1 & -1 \\ -1 & 5 & -1 \\ -1 & -1 & 5 \\ \sqrt{3} & \sqrt{3} & \sqrt{3} \end{bmatrix}, $$

and

$$ \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \\ \omega_4 \end{bmatrix} = A^{+}\,{}^{B}h . $$

The following MATLAB script computes these values along with two sanity checks on the results (as documented in the comments). It also sets up parameters for use in the HW8 Simulink model.

```
% Set up gyrostat with 4 RWAs
%
% HW 8
%
% Mass Properties
Ic=[3000 0 -30; 0 4000 20; -30 20 5000];
Id=100*eye(3);
invId=inv(Id);
Ibw=Ic-Id; % not needed for this calculation; just FYI
invIbw=inv(Ibw);
Iwi=70/(6000*pi/30)
%
% Kane damping
c=100;
%
% Initial conditions (per problem statement)
[V,D]=eig(Ic);
p3=V(:,3)
wBN0=0.5*p3;
wDN0=wBN0;
q0=[0 0 0 1]';
%
% Quadratic solution for W (negative root is for backward spin):
aa=Ic(1,3)^2+Ic(2,3)^2-Ic(3,3)^2;
bb=2*(D(3,3)*0.5*[-Ic(1,3) -Ic(2,3) 0]*V(:,3));
cc=0.25*D(3,3)^2;
W=(-bb-sqrt(bb^2-4*aa*cc))/(2*aa);
%
% Momentum for dynamic balance
h=W*[-Ic(1,3) -Ic(2,3) 0]'
%
%Sanity Check (final |H| = initial |H| within calculation tolerance)
H0=Ic*wBN0+h;
Hf=Ic*[0 0 W]'+h;
norm(Hf)-norm(H0)
%
% Given wheel Jacobian
A=[eye(3) sqrt(3)/3*[1 1 1]' ]*Iwi;
pinvA=pinv(A);
%
% Wheel speeds for dynamic balance
ww0=pinvA*h;
%
% Sanity check: (A*Iwi*Ww = h within calculation tolerance)
norm(A*ww0-h)
%
% Wheel limits (not part of HW 8 but maybe useful in HW 9)
wmax=6000*pi/30;      % 6000 rpm max rate
accelmax=1/Iwi;        % 1 Nm max torque
```

Output

```
Iwi =
   0.11140846016433
p3 =
  -0.01498895488327
   0.01997679194273
   0.99968808086082
W =
   0.50001301758127
h =
  15.00039052743805
 -10.00026035162537
                  0
ans =
   -9.094947017729282e-013
Ww =
  1.0e+002 *
   1.27163013268828
  -0.97242304264398
  -0.07480177251107
   0.12956047048539
ans =
    3.973605926116015e-015
```