MAE 606
Homework #5

1. When manipulating matrix expressions that derive from vectors and dyadics, we almost always need the matrices to consist of components in a single set of basis vectors. For this reason, we represent $\boldsymbol{r}_s$ in a general way that allows us to find the solar-panel properties in $\boldsymbol{b}$ axes. One can also resolve everything into the $\boldsymbol{s}$ axes, but doing so does not make much sense from an engineering standpoint because we care more about the body-fixed manufacturing and operational axes than we do about these moving solar-panel axes.

   $\boldsymbol{r}_s = \boldsymbol{r}_j + 1\boldsymbol{s}_3$

   Before deployment, this result yields

$$\boldsymbol{r}_s = 1\boldsymbol{b}_2 + 1\boldsymbol{b}_3 \Rightarrow {}^{\mathrm{B}}r_s = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}.$$

   After deployment, when $\begin{bmatrix} \boldsymbol{s}_1 \\ \boldsymbol{s}_2 \\ \boldsymbol{s}_3 \end{bmatrix} = {}^{\mathrm{S}}Q^{\mathrm{B}} \begin{bmatrix} \boldsymbol{b}_1 \\ \boldsymbol{b}_2 \\ \boldsymbol{b}_3 \end{bmatrix}$ and ${}^{\mathrm{S}}Q^{\mathrm{B}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}$, the expression has become

$$\boldsymbol{r}_s' = 1\boldsymbol{b}_2 + 1\boldsymbol{b}_2 \Rightarrow {}^{\mathrm{B}}r_s = \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}.$$

   Using these expressions in the familiar formula for the mass center of a collection of particles results in

$$m_{tot} = m_b + m_s$$

$$\boldsymbol{r}_{tot} = \frac{m_b \boldsymbol{r}_b + m_s \boldsymbol{r}_s}{m_{tot}}$$

$$\boldsymbol{r}_{tot} = \frac{5000(1.2\boldsymbol{b}_3) + 50(1\boldsymbol{b}_2 + 1\boldsymbol{b}_3)}{5050}$$

$$\boldsymbol{r}_{tot} = \frac{1}{101}\boldsymbol{b}_2 + \frac{121}{101}\boldsymbol{b}_3$$

   before deployment.

2. After deployment, the result is

$$\boldsymbol{r}_{tot}^{'} = \frac{5000(1.2\boldsymbol{b}_3) + 50(2\boldsymbol{b}_2)}{5050}$$

$$\boldsymbol{r}_{tot}^{'} = \frac{2}{101}\boldsymbol{b}_2 + \frac{120}{101}\boldsymbol{b}_3$$

3. The inertia matrix of the stowed solar panel about the panel mass center is

$$\boldsymbol{I}_s = [\boldsymbol{s}_1 \quad \boldsymbol{s}_2 \quad \boldsymbol{s}_3]\begin{bmatrix} 70 & 0 & -2 \\ 0 & 60 & 2 \\ -2 & 2 & 20 \end{bmatrix}\begin{bmatrix} \boldsymbol{s}_1 \\ \boldsymbol{s}_2 \\ \boldsymbol{s}_3 \end{bmatrix}$$

$$\boldsymbol{I}_s = [\boldsymbol{b}_1 \quad \boldsymbol{b}_2 \quad \boldsymbol{b}_3]\begin{bmatrix} 70 & 0 & -2 \\ 0 & 60 & 2 \\ -2 & 2 & 20 \end{bmatrix}\begin{bmatrix} \boldsymbol{b}_1 \\ \boldsymbol{b}_2 \\ \boldsymbol{b}_3 \end{bmatrix}.$$

Using the parallel-axis theorem to add the body and the panel inertia matrices together consists of

$$I_{tot} = {}^{B}I_b - m_b (r_b - r_{tot})^{\times}(r_b - r_{tot})^{\times} + {}^{B}I_s - m_s (r_s - r_{tot})^{\times}(r_s - r_{tot})^{\times}$$

$$I_{tot} = \begin{bmatrix} 5121.49 & 0 & -2.00 \\ 0 & 4061.98 & 61.90 \\ -2.00 & 61.90 & 3069.50 \end{bmatrix}$$

before deployment.

4. After deployment the panel inertia dyadic is

$$\boldsymbol{I}_s = [\boldsymbol{s}_1 \quad \boldsymbol{s}_2 \quad \boldsymbol{s}_3] \begin{bmatrix} 70 & 0 & -2 \\ 0 & 60 & 2 \\ -2 & 2 & 20 \end{bmatrix} \begin{bmatrix} \boldsymbol{s}_1 \\ \boldsymbol{s}_2 \\ \boldsymbol{s}_3 \end{bmatrix}$$

$$\boldsymbol{I}_s' = [\boldsymbol{b}_1 \quad \boldsymbol{b}_2 \quad \boldsymbol{b}_3]\, {}^{B}Q^{S} \begin{bmatrix} 70 & 0 & -2 \\ 0 & 60 & 2 \\ -2 & 2 & 20 \end{bmatrix} {}^{S}Q^{B} \begin{bmatrix} \boldsymbol{b}_1 \\ \boldsymbol{b}_2 \\ \boldsymbol{b}_3 \end{bmatrix}$$

$$\boldsymbol{I}_s' = [\boldsymbol{b}_1 \quad \boldsymbol{b}_2 \quad \boldsymbol{b}_3] \begin{bmatrix} 70 & -2 & 0 \\ -2 & 20 & -2 \\ 0 & -2 & 60 \end{bmatrix} \begin{bmatrix} \boldsymbol{b}_1 \\ \boldsymbol{b}_2 \\ \boldsymbol{b}_3 \end{bmatrix}$$

Then, for the system,

$$I_{tot}' = {}^{B}I_b - m_b \left(r_b - r_{tot}'\right)^{\times} \left(r_b - r_{tot}'\right)^{\times} + {}^{B}I_s' - m_s \left(r_s' - r_{tot}'\right)^{\times} \left(r_s' - r_{tot}'\right)^{\times}$$

$$I_{tot}' = \begin{bmatrix} 5339.31 & -2.00 & 0 \\ -2.00 & 4091.29 & 166.81 \\ 0 & 166.81 & 3258.02 \end{bmatrix}$$

5. One can compute the exact wobble angle by taking the eigenvectors of the inertia matrix, finding the eigenvector closest to ${}^{B}b_3$ (i.e. $[0\ 0\ 1]^T$), and computing the arccosine of the third component. However, the problem asks only for an approximate result. The approximation offered in the lecture depends on small-angle rotations about each of three axes. For the 3 axis,

$$\theta_w \approx \left[\left(\frac{I_{13}}{I_{11} - I_{33}} \right)^2 + \left(\frac{I_{23}}{I_{33} - I_{22}} \right)^2 \right]^{\frac{1}{2}}$$

$$\theta_w \approx \left[\left(\frac{61.9}{5121.49 - 3069.50} \right)^2 + \left(\frac{-2}{3069.50 - 4061.98} \right)^2 \right]^{\frac{1}{2}}$$

$$\theta_w \approx 0.0624 \text{ rad} \approx 3.57 \text{ deg}$$

6. The condition referred to here is spin about the maximum axis—i.e. the principal axis associated with the maximum moment of inertia. Taking the eigenvalues and eigenvectors of the stowed I_{tot} yields three principal moments of inertia,

   I_1=5121.49, I_2=4065.83, and I_3=3065.66.

   Associated with each of these principal moments of inertia is a particular eigenvector. In this case (to seven decimal places),

$$ {}^{B}p_1 = \begin{bmatrix} 0.9999995 \\ -0.0000570 \\ -0.0009764 \end{bmatrix},\quad {}^{B}p_2 = \begin{bmatrix} 0.0001175 \\ 0.9980755 \\ 0.0620098 \end{bmatrix},\quad {}^{B}p_3 = \begin{bmatrix} 0.0009709 \\ -0.0620098 \\ 0.9980750 \end{bmatrix}. $$

   Incidentally, MATLAB will not bother to order these eigenvectors in a convenient way. First of all, MATLAB likes to sort the eigenvalues so that they appear in descending order on the diagonal of the matrix it spits out. They don't necessarily correspond to the original body-axes ordering. It's up to you to notice what body axis is closest and choose the appropriate one. Also, although MATLAB's eigenvectors are normalized, they are arbitrary in sign. That is, you never know whether the eigenvector of interest will be closer to $\boldsymbol{b}_i$ or–$\boldsymbol{b}_i$. Once again, it's up to you to interpret the results correctly. It is best to re-order MATLAB's eigenvectors as shown here and scale them so that they readily suggest alignment with the positive $\boldsymbol{b}_i$ axes, keeping in mind that the resulting 3x3 matrix should be proper (det=+1) orthogonal.

   In this case, the eigenvector associated with the maximum axis is

$$ {}^{B}p_1 = \begin{bmatrix} 0.9999995 \\ -0.0000570 \\ -0.0009764 \end{bmatrix}. $$

   We note that the wobble angle we would compute from this result is very close to what the approximate formula gave in problem 5.

7. The procedure for the deployed system is identical. Here,

   I'_1=5339.31, I'_2=4123.44, and I'_3=3225.87.

   The eigenvector associated with I'_1 is

$$ {}^{B}p'_1 = \begin{bmatrix} 0.9999987 \\ -0.0016199 \\ -0.0001298 \end{bmatrix} $$

   Here, we note that the principal axis and the inertia matrix for the deployed system differ from the stowed values; however, the low mass and relatively good balance of the deployed body do not result in a qualitative change in the mass properties. For example, the maximum axis does not shift from $\boldsymbol{p}_1$ to $\boldsymbol{p}_3$, or anything as disruptive to the spacecraft design as that.

8. Now the relevance of H=40 Nms becomes clear. In the stable relative equilibrium, the spacecraft spins about the $\boldsymbol{p}_1$ axis. The inertia for that axis before deployment is I_1=5121.49 kg-m^2. In general, the angular velocity is

   $$\boldsymbol{\omega} = \boldsymbol{I}^{-1} \cdot \boldsymbol{H} \quad .$$

   Here, we know the angular velocity lies along the $\boldsymbol{p}_1$ axis because that's the stable relative equilibrium. Furthermore, we know its magnitude is simply $\dfrac{H}{I_1}$ = 40/5121.49=0.0078102 rad/sec. The kinetic energy is

   $$T = \frac{1}{2} I_1 {}^{\mathrm{P}}\omega_1^{\;2}$$

   $$T = \frac{1}{2} I_3 \left(\frac{H}{I_1} \right)^2$$

   T=0.1562045 J. Note that 1 J = 1 Nm.

9. Using the same equation with the deployed I'_1 yields the kinetic energy when the stowed system reaches equilibrium:

   T=0.1498320 J.

   The spacecraft spins a little more slowly in equilibrium after the panel is deployed. This result makes intuitive sense; it's the same effect an ice skater achieves when he or she sticks out an arm. Pulling in mass speeds up the spin. Pushing mass outward slows it down.

   Accordingly, the kinetic energy is also lower in equilibrium after deployment. One can conclude that if the deployment mechanism required no power and dissipated no energy (i.e. if T is constant during the deployment), the spacecraft must now be nutating around the new principal axis because it initially had too much energy to be in equilibrium after deployment. On the other hand, if the spacecraft began spinning about the new equilibrium immediately following deployment, the deployment process must have somehow extracted kinetic energy (via friction at the deployment mechanism, or perhaps through structural deformation and damping in the solar panel).